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The Hamiltonian formalism of the DNLS equation with a nonvanished boundary value

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Abstract

The Hamiltonian formalism of the derivative nonlinear Schrödinger equation with a nonvanishing boundary value is developed by the standard procedure. The action-angle variables are given explicitly. At the end of this work, a Galileo transformation is introduced to ensure that the conservative quantities obtained satisfy the Hamiltonian equation.

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1. Introduction

From the general view point, the complete integrability of a nonlinear equation means that it describes a multi-periodic system, that is, a Hamiltonian system with action-angle variables as canonical conjugate variables [1]. In the case of a complex field equation, such as the nonlinear Schrödinger (NLS) equation, one can introduce field density and its canonical conjugate momentum density in usual sense. For a real equation, such as the KdV equation, to formulate the Hamiltonian formalism, an alternative form of Poisson bracket for real field densities at two points has been introduced [2, 3]. Furthermore, the time dependence of angle variables derived from its Poisson bracket with action variables must be the same as that derived from the inverse scattering transform, which has not been paid much attention to.

The derivative nonlinear Schrödinger (DNLS) equation was proposed to describe nonlinear Alfvén waves in plasma [4, 5]. In the case of a vanishing boundary, it was solved by the inverse scattering transform (IST) [6], or other approaches [7–9], and its complete integrability was shown by Kundu in [10] through the r–s matrix formalism. In the other case of a nonvanishing boundary, the DNLS equation was discussed by some authors in terms of the usual spectral parameter [11, 12]. The multi-value problem of square root appears, and then the Riemann surface has to be introduced, which leads to complexity and ambiguity in the derivation. As a result, the affin parameter is introduced to clarify the multi-value problem [13, 14, 17], based upon which the Hamiltonian formalism of the DNLS equation with a nonvanishing boundary condition can be formulated naturally. We perform it in this work, and the effect of the linear coordinate transformation on the Hamiltonian theory is shown.

2. Poisson bracket

The DNLS equation with a nonvanishing boundary condition (DNLS$^+$ equation) is generally expressed as [4, 5]

$$\mathrm{i}u_t - u_{xx} + \mathrm{i}(|u|^2u)_x = 0, \tag{1}$$

where u is complex and $|u| \to \rho$ in the limit of $x \to \pm\infty$. Now, a particular form of the Poisson bracket is proposed:

$$\{u(x), \overline{u(y)}\} = \tfrac{1}{2}\{\partial_x - \partial_y\}\delta(x-y). \tag{2}$$

For two quantities Q, R, the Poisson bracket is

$$\{Q, R\} = \iint \mathrm{d}x\,\mathrm{d}y \left(\frac{\delta Q}{\delta u(x)} \frac{\delta R}{\delta \overline{u(y)}} \{u(x), \overline{u(y)}\} + \frac{\delta Q}{\delta \overline{u(x)}} \frac{\delta R}{\delta u(y)} \{\overline{u(x)}, u(y)\} \right). \tag{3}$$

Integrating by part, equation (3) becomes

$$\{Q, R\} = -\frac{1}{2} \iint \mathrm{d}x\,\mathrm{d}y \left(\left\{\partial_x \frac{\delta Q}{\delta u(x)}\right\} \frac{\delta R}{\delta \overline{u(y)}} - \frac{\delta Q}{\delta u(x)} \left\{\partial_x \frac{\delta R}{\delta \overline{u(y)}}\right\} + \left\{\partial_x \frac{\delta Q}{\delta \overline{u(x)}}\right\} \frac{\delta R}{\delta u(y)} - \frac{\delta Q}{\delta \overline{u(x)}} \left\{\partial_x \frac{\delta R}{\delta u(y)}\right\} \right) \delta(x-y). \tag{4}$$

Thus, the Hamiltonian equation is obtained as

$$u_t(x) = \{H, u(x)\}, \qquad H = \int \mathrm{d}x\, \mathcal{H}(x), \qquad \mathcal{H}(x) = \frac{1}{2}|u|^4 - \mathrm{i}u_x\bar{u}. \tag{5}$$

3. Lax pair

The first of the Lax equations is

$$\partial_x F(x,t,\lambda) = L(x,t,\lambda)F(x,t,\lambda), \tag{6}$$

where λ is a spectral parameter. L is a 2×2 matrix

$$L = -\mathrm{i}\lambda^2\sigma_3 + \lambda U, \qquad U = u\sigma_+ + \bar{u}\sigma_-, \tag{7}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{8}$$

and ζ is an auxiliary parameter such that

$$\lambda = \tfrac{1}{2}(\zeta + \rho^2\zeta^{-1}), \qquad \kappa = \tfrac{1}{2}(\zeta - \rho^2\zeta^{-1}). \tag{9}$$

As the asymptotic free Jost solution is $E(x,\zeta) = (I + \rho\zeta\sigma_2)\,\mathrm{e}^{-\mathrm{i}\lambda\kappa x\sigma_3}$, we define the Jost solutions

$$\begin{aligned} \Psi(x,\zeta) &= (\tilde{\psi}(x,\zeta), \psi(x,\zeta)) \to E(x,\zeta), && \text{as} \quad x \to \infty, \\ \Phi(x,\zeta) &= (\phi(x,\zeta), \tilde{\phi}(x,\zeta)) \to Q^{-1}(\alpha)E(x,\zeta), && \text{as} \quad x \to -\infty, \end{aligned} \tag{10}$$

where $Q(\alpha) = \mathrm{e}^{\mathrm{i}\frac{1}{2}\alpha\sigma_3}$. Then the monodramy matrix $T(\lambda)$ is given by

$$T(\zeta) = \Psi^{-1}(x, \zeta)\Phi(x, \zeta), \qquad T(\zeta) = \begin{pmatrix} a(\zeta) & \tilde{b}(\zeta) \\ b(\zeta) & \tilde{a}(\zeta) \end{pmatrix}. \tag{11}$$

From (10), $a(\zeta)$ can be analytically continued into the first and the third quadrants and $\tilde{a}(\zeta)$ into the second and the fourth quadrants. The continuous spectrum is composed of real ζ^2, that is, composed of real ζ and imaginary ζ.

Similar to the NLS$^+$ equation, there are several reduction transformation properties:

$$\tilde{\psi}(x, \bar{\zeta}) = \sigma_1\psi(x, \zeta), \qquad \tilde{\phi}(x, \bar{\zeta}) = \sigma_1\phi(x, \zeta), \tag{12}$$

$$\tilde{a}(\bar{\zeta}) = a(\zeta), \qquad \tilde{b}(\bar{\zeta}) = b(\zeta), \tag{13}$$

and, under the transformation $\zeta \to \rho\zeta^{-1}$,

$$\tilde{\psi}(x, \rho^2\zeta^{-1}) = \mathrm{i}\rho^{-1}\zeta\psi(x, \zeta), \qquad \tilde{\phi}(x, \rho^2\zeta^{-1}) = -\mathrm{i}\rho^{-1}\zeta\phi(x, \zeta), \tag{14}$$

$$\tilde{a}(\rho^2\zeta^{-1}) = a(\zeta), \qquad \tilde{b}(\rho^2\zeta^{-1}) = -b(\zeta). \tag{15}$$

in which the second ones of (13) and (15) require $\lambda\kappa$ as real.

And there are some reduction properties of only the DNLS$^+$ equation; for example, since

$$L(-\zeta) = \sigma_3 L(\zeta)\sigma_3, \tag{16}$$

we have

$$T(-\zeta) = \sigma_3 T(\zeta)\sigma_3, \tag{17}$$

that is,

$$a(-\zeta) = a(\zeta), \qquad b(-\zeta) = -b(\zeta). \tag{18}$$

Since $a(\lambda)$ is assumed to have N simple poles in the first and third quadrants, $a(\lambda)$ can be expressed as [1]

$$a(\lambda) = \prod_{n=1}^{N} \frac{\zeta - \zeta_n}{\zeta - \bar{\zeta}_n}\frac{\bar{\zeta}_n}{\zeta_n} \exp\left\{\frac{\zeta}{\mathrm{i}2\pi}\int_\Gamma \frac{\ln|a(\zeta')|^2}{(\zeta' - \zeta)\zeta'}\right\}, \tag{19}$$

where the integral contour $\Gamma = (0, +\infty)\bigcup(0, -\infty)\bigcup(+\mathrm{i}\infty, 0)\bigcup(-\mathrm{i}\infty, 0)$ along the real and imaginary axes.

4. Poisson bracket for the monodramy matrix

Since $\partial_x \det\Psi(x, \zeta) = 0$ and $\partial_x \det\Phi(x, \zeta) = 0$, we have

$$\det\Psi(x, \zeta) = \det\Phi(x, \zeta) = \det E(x, \zeta) = 1 - \rho^2\zeta^{-2}, \tag{20}$$

and thus,

$$\det T(\zeta) = 1, \qquad a(\zeta)\tilde{a}(\zeta) - b(\zeta)\tilde{b}(\zeta) = 1, \tag{21}$$

$$T^{-1}(\zeta) = \begin{pmatrix} \tilde{a}(\zeta) & -\tilde{b}(\zeta) \\ -b(\zeta) & a(\zeta) \end{pmatrix} \tag{22}$$

and so on.

Introducing the usual direct product $\otimes$, the Poisson bracket of the monodramy matrix is defined as

$$\{T(\zeta) \overset{\otimes}{,} T^{-1}(\zeta')\}_{ij,kl} = \{T(\zeta)_{ij}, T^{-1}(\zeta')_{kl}\}. \tag{23}$$

Substituting equations (11)–(15), the explicit expression of equation (23) is

$$\{T(\zeta) \overset{\otimes}{,} T^{-1}(\zeta')\} = -\frac{1}{2}\int \mathrm{d}x\, \Psi^{-1}(x,\zeta)\Phi^{-1}(x,\zeta')R\Phi(x,\zeta)\Psi(x,\zeta'), \tag{24}$$

where

$$\begin{aligned} R = &-(\mathrm{i}2\lambda^3\sigma_+ - \lambda^2\bar{u}\sigma_3)\otimes(\lambda'\sigma_-) - (\lambda\sigma_+)\otimes(\mathrm{i}2\lambda'^3\sigma_- + \lambda'^2 u\sigma_3) \\ &+ (\mathrm{i}2\lambda^3\sigma_- + \lambda^2 u\sigma_3)\otimes(\lambda'\sigma_+) + (\lambda\sigma_-)\otimes(\mathrm{i}2\lambda'^3\sigma_+ - \lambda'^2\bar{u}\sigma_3). \end{aligned} \tag{25}$$

Equation (25) is expressed in a matrix with row $\{i'l'\}$ and column $\{j'm'\}$,

$$-\begin{pmatrix} 0 & -\lambda^2\lambda' u & \lambda\lambda'^2 u & 0 \\ -\lambda^2\lambda'\bar{u} & 0 & \mathrm{i}2\lambda^3\lambda' + \mathrm{i}2\lambda\lambda'^3 & -\lambda\lambda'^2 u \\ \lambda\lambda'^2\bar{u} & -\mathrm{i}2\lambda^3\lambda' - \mathrm{i}2\lambda\lambda'^3 & 0 & \lambda^2\lambda' u \\ 0 & -\lambda\lambda'^2\bar{u} & \lambda^2\lambda'\bar{u} & 0 \end{pmatrix}. \tag{26}$$

5. Another direct product for Poisson bracket

In the usual method to formulate the Hamiltonian theory, one considers [1]

$$\partial_x(\{\Psi^{-1}(\zeta)\Psi(\zeta')\}\otimes'\{\Phi^{-1}(\zeta')\Phi(\zeta)\}), \tag{27}$$

where another direct product $\otimes'$ is introduced,

$$A_{im}B_{lj} = (A\otimes' B)_{il,jm}. \tag{28}$$

From the first Lax equation, equation (27) becomes

$$\begin{aligned} &\Psi^{-1}(\zeta)\{L(\zeta') - L(\zeta)\}\Psi(\zeta')\otimes'\Phi^{-1}(\zeta')\Phi(\zeta) \\ &\quad + \Psi^{-1}(\zeta)\Psi(\zeta')\otimes'\Phi^{-1}(\zeta')\{L(\zeta) - L(\zeta')\}\Phi(\zeta), \end{aligned} \tag{29}$$

that is,

$$\Psi^{-1}(x,\zeta)\Phi^{-1}(x,\zeta')W_0\Phi(x,\zeta)\Psi(x,\zeta'), \tag{30}$$

where

$$\begin{aligned} W_0 = &\,\mathrm{i}(\lambda^2 - \lambda'^2)\{\sigma_3\otimes' I - I\otimes'\sigma_3\} - (\lambda-\lambda')u\{\sigma_+\otimes' I - I\otimes'\sigma_+\} \\ &- (\lambda-\lambda')\bar{u}\{\sigma_-\otimes' I - I\otimes'\sigma_-\}. \end{aligned} \tag{31}$$

Equation (31) may be written in the matrix form explicitly as

$$\begin{pmatrix} 0 & -(\lambda-\lambda')u & (\lambda-\lambda')u & 0 \\ (\lambda-\lambda')\bar{u} & 0 & \mathrm{i}2(\lambda^2-\lambda'^2) & -(\lambda-\lambda')u \\ -(\lambda-\lambda')\bar{u} & -\mathrm{i}2(\lambda^2-\lambda'^2) & 0 & (\lambda-\lambda')u \\ 0 & (\lambda-\lambda')\bar{u} & -(\lambda-\lambda')\bar{u} & 0 \end{pmatrix}. \tag{32}$$

It is obvious that equation (32) is not proportional to equation (26), which means that another expression is necessary to construct the Hamiltonian formalism.

Considering

$$\partial_x(\{\Psi^{-1}(\zeta)\sigma_3\Psi(\zeta')\}\otimes'\{\Phi^{-1}(\zeta')\sigma_3\Phi(\zeta)\}), \tag{33}$$

it is equal to

$$\begin{aligned} &\Psi^{-1}(\zeta)\{\sigma_3 L(\zeta') - L(\zeta)\sigma_3\}\Psi(\zeta')\otimes'\Phi^{-1}(\zeta')\sigma_3\Phi(\zeta) \\ &\quad + \Psi^{-1}(\zeta)\sigma_3\Psi(\zeta')\otimes'\Phi^{-1}(\zeta')\{\sigma_3 L(\zeta) - L(\zeta')\sigma_3\}\Phi(\zeta) \end{aligned} \tag{34}$$

also from the first Lax equation. Similar to (30), (34) may be written in the form

$$\Psi^{-1}(x,\zeta)\Phi^{-1}(x,\zeta')W_3\Phi(x,\zeta)\Psi(x,\zeta'), \tag{35}$$

where

$$\begin{aligned} W_3 = {} & \mathrm{i}(\lambda^2-\lambda'^2)\{I\otimes'\sigma_3-\sigma_3\otimes' I\}+(\lambda+\lambda')u\{\sigma_+\otimes'\sigma_3+\sigma_3\otimes'\sigma_+\} \\ & -(\lambda+\lambda')\bar{u}\{\sigma_-\otimes'\sigma_3+\sigma_3\otimes'\sigma_-\}, \end{aligned} \tag{36}$$

explicitly

$$\begin{pmatrix} 0 & (\lambda+\lambda')u & (\lambda+\lambda')u & 0 \\ -(\lambda+\lambda')\bar{u} & 0 & -\mathrm{i}2(\lambda^2-\lambda'^2) & -(\lambda+\lambda')u \\ -(\lambda+\lambda')\bar{u} & \mathrm{i}2(\lambda^2-\lambda'^2) & 0 & -(\lambda+\lambda')u \\ 0 & (\lambda+\lambda')\bar{u} & (\lambda+\lambda')\bar{u} & 0. \end{pmatrix}. \tag{37}$$

Defining

$$\Delta_\alpha \equiv \lim_{L\to\infty}\Psi^{-1}(x,\zeta)\sigma_\alpha\Psi(x,\zeta')\otimes'\Phi^{-1}(x,\zeta')\sigma_\alpha\Phi(x,\zeta)|_{x=-L}^{x=L}, \tag{38}$$

there should be

$$\{T(\zeta)\overset{\otimes}{,}T^{-1}(\zeta')\} = f_0\Delta_0+f_3\Delta_3, \tag{39}$$

where two constant coefficients f_0 and f_3 are introduced, that is, $f_0(32)+f_3(37)=(26)$. A comparison between the corresponding elements of matrices in two sides yields

$$-(\lambda-\lambda')f_0+(\lambda+\lambda')f_3=\lambda^2\lambda', \qquad (\lambda-\lambda')f_0+(\lambda+\lambda')f_3=-\lambda\lambda'^2. \tag{40}$$

It is found that

$$f_0=-\frac{1}{2}\lambda\lambda'\frac{\lambda+\lambda'}{\lambda-\lambda'}, \qquad f_3=\frac{1}{2}\lambda\lambda'\frac{\lambda-\lambda'}{\lambda+\lambda'}. \tag{41}$$

6. Explicit expression of Poisson bracket of the monodramy matrix

From (39)–(41), the Poisson bracket $\{T(\lambda)\overset{\otimes}{,}T^{-1}(\lambda')\}$, i.e.

$$\begin{pmatrix} \{a,\tilde{a}'\} & -\{a,\tilde{b}'\} & \{\tilde{b},\tilde{a}'\} & -\{\tilde{b},\tilde{b}'\} \\ -\{a,b'\} & \{a,a'\} & -\{\tilde{b},b'\} & \{\tilde{b},a'\} \\ \{b,\tilde{a}'\} & -\{b,\tilde{b}'\} & \{\tilde{a},\tilde{a}'\} & -\{\tilde{a},\tilde{b}'\} \\ -\{b,b'\} & \{b,a'\} & -\{\tilde{a},b'\} & \{\tilde{a},a'\} \end{pmatrix} \tag{42}$$

is equal to the matrix $f_0\Delta_0+f_3\Delta_3$, for example,

$$\{a,b'\}=\frac{1}{2}\lambda\lambda'\left\{\frac{\lambda\lambda'-\rho^2}{\kappa\kappa'}\frac{\lambda+\lambda'}{\lambda-\lambda'+\mathrm{i}0}+\frac{\lambda\lambda'+\rho^2}{\kappa\kappa'}\frac{\lambda-\lambda'}{\lambda+\lambda'+\mathrm{i}0}\right\}ab' \tag{43}$$

and

$$\{\tilde{a},b'\}=-\frac{1}{2}\lambda\lambda'\left\{\frac{\lambda\lambda'-\rho^2}{\kappa\kappa'}\frac{\lambda+\lambda'}{\lambda-\lambda'-\mathrm{i}0}+\frac{\lambda\lambda'+\rho^2}{\kappa\kappa'}\frac{\lambda-\lambda'}{\lambda+\lambda'-\mathrm{i}0}\right\}ab'. \tag{44}$$

As ζ is pure imaginary, i.e. $\zeta=\mathrm{i}\eta$, noting that $\delta(\mathrm{i}\eta)=\delta(\eta)$, we also have the same results with (43) and (44).

7. Action-angle variables in a continuous spectrum

From the inverse scattering transform, $a(\zeta)$ and $\tilde{a}(\zeta)$ are independent of t, and $b(\zeta)$ and $\tilde{b}(\zeta)$ depend on t. And noting the reduction transformation properties (15) and (18), we may restrict ourselves to consider $\zeta, \zeta' > \rho$. Thus, from (43) and (44), there is

$$\{|a(\zeta)|^2, b(\zeta')\} = -\mathrm{i}2\lambda^3\pi|a(\zeta)|^2 b(\zeta)(1-\rho^2\zeta^{-2})^{-1}\delta(\zeta-\zeta'). \tag{45}$$

The angle variable $Q(\zeta)$ and action variable $P(\zeta)$ are chosen, respectively, to be

$$Q(\zeta) = \arg b(\zeta) = \frac{1}{2\mathrm{i}}\ln\frac{b(\zeta)}{\bar{b}(\zeta)}, \qquad P(\zeta) = F(|a(\zeta)|^2), \tag{46}$$

such that

$$\{P(\zeta), Q(\zeta')\} = -\delta(\zeta-\zeta'), \tag{47}$$

where the unknown function F is to be determined by (45).

Substituting (45) into (47), it is easy to find

$$F'(|a(\zeta)|^2)2\lambda^3\pi|a(\zeta)|^2(1-\rho^2\zeta^{-2})^{-1} = 1. \tag{48}$$

Thus, the action variable $P(\zeta)$ is chosen as

$$P(\zeta) = F(|a(\zeta)|^2) = \frac{1-\rho^2\zeta^{-2}}{2\lambda^3\pi}\ln|a(\zeta)|^2. \tag{49}$$

8. Action-angle variable in a discrete spectrum

From the inverse scattering transform we know that λ_n, zero of $a(\lambda)$, is independent of time and b_n is dependent on time periodically. Hence, we need the Poisson bracket of ζ_m with b_n, and of them with $a(\zeta)$ and $b(\zeta)$. From (19) and (45), we obtain

$$\{\ln\breve{a}(\zeta), b(\zeta')\} + \sum_m\left(\frac{\{\bar{\zeta}_m, b(\zeta')\}}{\zeta-\bar{\zeta}_m} - \frac{\{\zeta_m, b(\zeta')\}}{\zeta-\zeta_m}\right)$$
$$= b(\zeta')\left\{\lambda\lambda'\frac{\lambda+\lambda'}{2}\frac{\lambda\lambda'-\rho^2}{\kappa\kappa'}\frac{1}{\lambda-\lambda'+\mathrm{i}0} + \lambda\lambda'\frac{\lambda-\lambda'}{2}\frac{\lambda\lambda'+\rho^2}{\kappa\kappa'}\frac{1}{\lambda+\lambda'+\mathrm{i}0}\right\}. \tag{50}$$

If $\zeta = \zeta_m$, then $\lambda_m = \operatorname{Re}\zeta_m$ is real, $\lambda_m - \lambda' + \mathrm{i}0 \neq 0$ and $\lambda_m + \lambda' + \mathrm{i}0 \neq 0$ since λ' is real. The right-hand side indicates that λ_m is not a pole, that is, $\{\zeta_m, b(\zeta')\} = 0$. Similarly, we have $\{\bar{\zeta}_m, b(\zeta')\} = 0$. Then by the standard procedure, similar to (50), we have

$$\{\ln\breve{a}(\zeta), b_n\} + \sum_m\left(\frac{\{\bar{\zeta}_m, b_n\}}{\zeta-\bar{\zeta}_m} - \frac{\{\zeta_m, b_n\}}{\zeta-\zeta_m}\right)$$
$$= b_n\left\{\lambda\lambda_n\frac{\lambda+\lambda_n}{2}\frac{\lambda\lambda_n-\rho^2}{\kappa\kappa_n}\frac{1}{\lambda-\lambda_n} + \lambda\lambda_n\frac{\lambda-\lambda_n}{2}\frac{\lambda\lambda_n+\rho^2}{\kappa\kappa_n}\frac{1}{\lambda+\lambda_n}\right\}. \tag{51}$$

The right-hand side has a pole at $\zeta = \zeta_n$; noting that $\lambda - \lambda_n = \frac{1}{2}(\zeta-\zeta_n)\left(1-\rho^2\zeta^{-1}\zeta_n^{-1}\right)$, we obtain

$$\{\zeta_m, b_n\} = -2\lambda_n^3\left(1-\rho^2\zeta_n^2\right)^{-1}b_n\delta_{mn}. \tag{52}$$

This result is similar to that of the Hamiltonian theory for other nonlinear equations. As $\bar{\zeta}_n \neq \rho^2\zeta_n^{-1}$ for the DNLS$^+$ equation, it is important that $\zeta - \bar{\zeta}_n$ is not a factor of $\lambda - \lambda_n$, that is

$$\{\bar{\zeta}_m, b_n\} = 0. \tag{53}$$

Furthermore, we also have $\{a(\zeta), \zeta_m\} = 0$ and $\{\zeta_n, \zeta_m\} = 0$.

In the discrete spectrum case, the angle variable is

$$Q_n = \ln b_n \tag{54}$$

and the action variable is assumed to be $P_m = G(\zeta_m)$, where G is an unknown function. Their Poisson bracket must be $\{P_m, Q_n\} = -\delta_{mn}$, and then, noting (52) and (54), we have

$$G'(\zeta_m) 2\lambda_m^3 \left(1 - \rho^2 \zeta_m^{-2}\right)^{-1} = 1. \tag{55}$$

We thus obtain

$$P_m = G(\zeta_m) = -\frac{1}{2\lambda_m^2}. \tag{56}$$

9. Conservative quantities

Since the first one of the Lax pair of the DNLS$^+$ equation is the same as that of the NLS$^+$ equation, the conservative quantities are the same. We have

$$\ln a(\zeta) = \sum_n \ln\left(\frac{\zeta - \zeta_n}{\zeta - \bar{\zeta}_n} \frac{\bar{\zeta}_n}{\zeta_n}\right) - \frac{\zeta}{\mathrm{i}2\pi} \int_\Gamma \mathrm{d}\zeta' \frac{\ln|a(\zeta')|^2}{(\zeta' - \zeta)\zeta'}. \tag{57}$$

Since $a(\zeta)$ is a constant in time, all terms in the expansion of $|\zeta| \to \infty$ are constant, for example,

$$I_0 = \sum_n 2 \ln \frac{\bar{\zeta}_n}{\zeta_n} + \frac{1}{\mathrm{i}\pi} \int_{\Gamma_+} \mathrm{d}\zeta' \frac{1}{\zeta'} \ln|a(\zeta')|^2, \tag{58}$$

$$I_2 = \sum_m \left(\bar{\zeta}_m^2 - \zeta_m^2\right) + \frac{1}{\mathrm{i}\pi} \int_{\Gamma_+} \mathrm{d}\zeta' \zeta' \ln|a(\zeta')|^2, \tag{59}$$

etc, where we have taken account of $|a(\zeta)|^2 = |a(-\zeta)|^2$ and the condition that $-\zeta_m$ is a zero of $a(\zeta)$ as long as ζ_m is a zero of $a(\zeta)$, see (19). The Hamiltonian is assumed to be

$$H = \mathrm{i}I_2 - \mathrm{i}\eta I_0 = \sum_m \mathrm{i}\left[\left(\bar{\zeta}_m^2 - \zeta_m^2\right) - 2\eta(\ln \bar{\zeta}_m - \ln \zeta_n)\right] + \frac{1}{\pi} \int_{\Gamma_+} \mathrm{d}\zeta' \left(\zeta' - \eta \frac{1}{\zeta'}\right) \ln|a(\zeta')|^2, \tag{60}$$

where the contour Γ_+ is along the first quadrant, and η is a real constant we shall determine. The integral domain $(0, \rho)$ can be transformed to (ρ, ∞) by $\zeta' \to \rho^2 \zeta'^{-1}$; the integral part is now given by

$$H_{\mathrm{int}} = \frac{1}{\mathrm{i}\pi} \int_{\Gamma_+ \cup \{|\zeta'| > \rho\}} \mathrm{d}\zeta' \{(\zeta' + \rho^4 \zeta'^{-3}) - 2\eta \zeta'^{-1}\} \ln|a(\zeta')|^2. \tag{61}$$

From (45), H_{int} must involve a factor $(1 - \rho^2 \zeta'^{-2})$; it is easily seen that if and only if $\eta = \rho^2$, the factor in bracket can be factored as

$$(\zeta' + \rho^4 \zeta'^{-3}) - 2\eta \zeta'^{-1} = (\zeta' - \rho^2 \zeta'^{-1})(1 - \rho^2 \zeta'^{-2}). \tag{62}$$

In this choice, we obtain from (45)

$$\{H_{\mathrm{int}}, b(\zeta)\} = -\mathrm{i}4\lambda^3 \kappa b(\zeta). \tag{63}$$

From equations (52) and (53), we have

$$\{H_{\mathrm{dis}}, b_n\} = -\mathrm{i}4\lambda_n^3 \kappa_n b_n, \tag{64}$$

where H_{dis} is the summation part. From equations (63) and (64), there are

$$b(t,\zeta) = b(0,\zeta)\,\mathrm{e}^{\mathrm{i}4\lambda^3\kappa t}, \qquad b_n(t) = b_n(0)\,\mathrm{e}^{\mathrm{i}4\lambda_n^3\kappa_n t}. \tag{65}$$

In such a choice, the second one of the Lax pair should be

$$M = -\mathrm{i}2\lambda^4\sigma_3 + 2\lambda^3 U - \mathrm{i}\lambda^2(U^2-\rho^2)\sigma_3 + \lambda U(U^2-\rho^2) - \mathrm{i}\lambda U_x\sigma_3, \tag{66}$$

which is different from the usual form. As a result, the compatibility condition gives

$$\mathrm{i}u_t - u_{xx} + \mathrm{i}[(|u|^2-\rho^2)u]_x = 0, \tag{67}$$

which differs from the usual form of the DNLS$^+$ equation (1) by a Galileo transformation

$$t' = t, \qquad x' = x - \rho^2 t. \tag{68}$$

Correspondingly, different from equation (5), here the Hamiltonian density is chosen as

$$\mathcal{H}(x) = \tfrac{1}{2}(|u|^2-\rho^2)^2 - \mathrm{i}u_x\bar{u}. \tag{69}$$

Thus, we formulate the complete Hamiltonian theory for the DNLS equation with nonvanishing boundary conditions. At the end, the linear Galileo transformation is introduced to take the time dependence of angle variables derived from its Poisson bracket with the Hamiltonian compatible with that derived from the second Lax equation.

As u represents the complex transverse magnetic field, the DNLS equation with nonvanishing boundary conditions describes magnetohydrodynamic waves in plasma propagating in an arbitrary angle to the ambient magnetic field, while the vanishing boundary conditions can only deal with waves exactly parallel to the ambient field [4, 5, 13–16]. In the case of a vanishing boundary, the DNLS equation was solved by the inverse scattering transform (IST) [6], or other approaches [7–9]. However, the DNLS equation with a nonvanishing boundary has never been solved exactly though some works have tried to do it [9, 11–14, 17]. Being one of a few well-known unsolved completely integrable nonlinear evolution equations, it is worth trying our best to find the final solution for the DNLS$^+$ equation. The success of formulating the Hamiltonian theory should be greatly beneficial to the procedure of solving the DNLS equation.

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